Chapter 2 Matrix Algebra

Section 2.1 Matrix Operations

Definitions

- The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$
  - An example is the $n \times n$ **identity matrix**, I_n.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \ldots$$

- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as $\mathbf{0}$.

Sums and Scalar Multiples

1. If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.
2. If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.

Theorem 1 Let A, B, and C be matrices of the same size, and let r and s be scalars.

- $A + B = B + A$
- $r(A + B) = rA + rB$
- $(A + B) + C = A + (B + C)$
- $(r + s)A = rA + sA$
- $A + \mathbf{0} = A$
- $r(sA) = (rs)A$

Matrix Multiplication

Motivation

When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\mathbf{x})$.

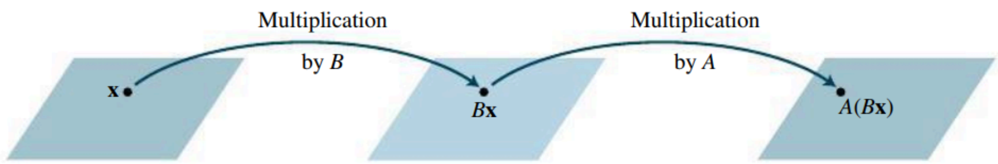

FIGURE 2 Multiplication by B and then A.

Thus $A(B\mathbf{x})$ is produced from $\mathbf{x}$ by a composition of mappings-the linear transformations studied in Section 1.8. We want to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that

$$A(B\mathbf{x}) = (AB)\mathbf{x}$$

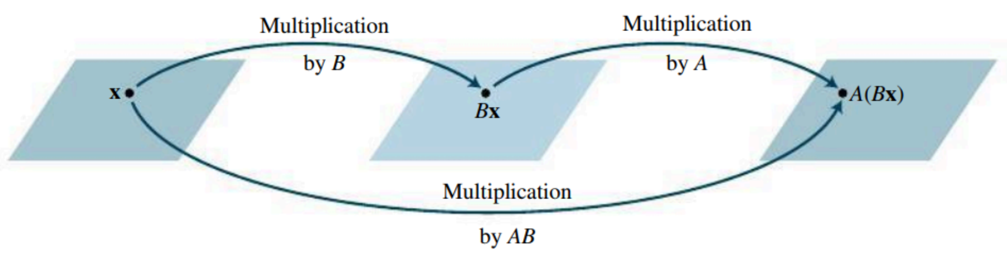

FIGURE 3 Multiplication by AB.

Definition (Matrix Multiplication) If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is,

$$AB = A\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p\end{bmatrix} = \begin{bmatrix}A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p\end{bmatrix}$$

Remark:

- Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B.
- AB has the same number of rows as A and the same number of columns as B.

$$A_{m\times n} \cdot B_{n\times p} = (AB)_{m\times p}$$

🤔**Question.** If a matrix A is 5×6 and the product AB is 5×8, what is the size of B? 6×8

$$A_{5\times 6} \cdot B_{6\times 8} = (AB)_{5\times 8}$$

Example 1. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}$,
$C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}$, $E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

Compute $A + 2B, 3C - E, CB, EB$. If an expression is undefined, explain why.

- $A+2B = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -3 & 2 \end{bmatrix} + 2 \cdot \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 16 & -10 & 1 \\ 6 & -11 & -4 \end{bmatrix}$

- $3C - E$ is undefined since $3C$ is 2×2, but E is 2×1

- $$C_{2\times 2} \cdot B_{2\times 3} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \cdot \begin{array}{c} \begin{matrix} \vec{b}_1 \downarrow & \vec{b}_2 \downarrow & \vec{b}_3 \downarrow \end{matrix} \\ \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} \end{array} = \begin{bmatrix} C\vec{b}_1 & C\vec{b}_2 & C\vec{b}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 7 + 2\cdot 1 & 1\cdot(-5) + 2\cdot(-4) & 1\cdot 1 + 2\cdot(-3) \\ -2\cdot 7 + 1\cdot 1 & (-2)\cdot(-5) + 1\cdot(-4) & -2\cdot 1 + 1\cdot(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -13 & -5 \\ -13 & 6 & -5 \end{bmatrix}$$

- $E_{2\times 1} B_{2\times 3}$ is not defined because the number of columns of E is not the same as the number of rows of B.

ROW-COLUMN RULE FOR COMPUTING AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Example 2. Compute the product AB in two ways:

(a) by the definition, where $A\mathbf{b}_1$ and $A\mathbf{b}_2$ are computed separately, and

(b) by the row-column rule for computing AB.

$$A = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}, \quad B = \begin{array}{c} \begin{array}{cc} \vec{b}_1 & \vec{b}_2 \end{array} \\ \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} \end{array}$$

ANS: (a). $A\vec{b}_1 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \\ 23 \end{bmatrix}$, $A\vec{b}_2 = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 14 \\ -9 \\ 4 \end{bmatrix}$

$$AB = [A\vec{b}_1 \quad A\vec{b}_2] = \begin{bmatrix} -4 & 14 \\ -3 & -9 \\ 23 & 4 \end{bmatrix}$$

(b) Row-column Rule:

$$AB = \begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4\cdot 1 - 2\cdot 4 & 4\cdot 3 - 2\cdot(-1) \\ -3\cdot 1 + 0\cdot 4 & -3\cdot 3 - 0\cdot(-1) \\ 3\cdot 1 + 5\cdot 4 & 3\cdot 3 + 5\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 14 \\ -3 & -9 \\ 23 & 4 \end{bmatrix}$$

Example 3. If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and third columns of B.

ANS: $AB = [A\vec{b}_1 \quad A\vec{b}_2 \quad A\vec{b}_3] = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$

The 1st column of B satisfies the equation

$$A\vec{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 6 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & 4 \end{array}\right]. \text{ So } \vec{b}_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

Similarly.

$$A\vec{x} = A\vec{b}_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -2 & 5 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

So $\vec{b}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Properties of Matrix Multiplication

Theorem 2. Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B+C) = AB + AC$ (left distributive law)
- $(B+C)A = BA + CA$ (right distributive law)
- $r(AB) = (rA)B = A(rB)$ for any scalar r
- $I_m A = A = AI_n$ (identity for matrix multiplication)

Warnings:

1. In general, $AB \neq BA$.
2. The cancellation laws do not hold for matrix multiplication. That is, if $AB = AC$, then it is not true in general that $B = C$. (See Exercise 10.)
3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$. (See Exercise 12.)

Power of a Matrix

If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A.

$$A^k = \underbrace{A \cdots A}_{k}$$

The Transpose of a Matrix

Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Example: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times 2}$, $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$

Theorem 3. Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $\left(A^T\right)^T = A$
- $(A+B)^T = A^T + B^T$
- For any scalar r, $(rA)^T = rA^T$
- $(AB)^T = B^T A^T$ (The transpose of a product of matrices equals the product of their transposes in the reverse order.)

Example 4. Let $\mathbf{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Compute $\mathbf{u}^T\mathbf{v}$, $\mathbf{v}^T\mathbf{u}$, $\mathbf{u}\mathbf{v}^T$, and $\mathbf{v}\mathbf{u}^T$. How are they related?

ANS: $\vec{u}^T \vec{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix}_{1\times 3} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3\times 1} = -2a + 3b - 4c$

$\parallel$

$\vec{v}^T \vec{u} = \begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3} \cdot \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}_{3\times 1} = -2a + 3b - 4c$

$\vec{u}\vec{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}_{3\times 1} \cdot \begin{bmatrix} a & b & c \end{bmatrix}_{1\times 3} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}_{3\times 3}$

$$\vec{v}\,\vec{u}^T = \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{3\times 1} \underbrace{\begin{bmatrix} -2 & 3 & -4 \end{bmatrix}}_{1\times 3} = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

Notice : • $\vec{u}^T\vec{v}$ and $\vec{v}^T\vec{u}$ are real numbers.
and real numbers equal to their transposes.

• $\vec{u}\,\vec{v}^T = (\vec{v}\,\vec{u}^T)^T$

You need to show this in the the Handwritten HW.

Hint : Use Thm 3.

The following questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Compute AD and DA. Explain how the columns or rows of A change when A is multiplied by D on the right or on the left. Find a 3×3 matrix B, not the identity matrix or the zero matrix, such that $AB = BA$.

Solution.

$$AD = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 8 & 9 \\ 2 & 16 & 15 \end{bmatrix}$$

$$DA = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 8 & 12 \\ 3 & 12 & 15 \end{bmatrix}$$

Right-multiplication (that is, multiplication on the right) by the diagonal matrix D multiplies each column of A by the corresponding diagonal entry of D.

Left-multiplication by D multiplies each row of A by the corresponding diagonal entry of D.

To make $AB = BA$, one can take B to be a multiple of I_3. For instance, if $B = 3I_3$, then AB and BA are both the same as $3A$.

Exercise 6. Let $A = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$. Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

Solution. Consider $B = [\mathbf{b}_1 \quad \mathbf{b}_2]$. To make $AB = 0$, one needs $A\mathbf{b}_1 = \mathbf{0}$ and $A\mathbf{b}_2 = \mathbf{0}$.

By inspection of A, a suitable $\mathbf{b}_1$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, or any multiple of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Example: $B = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$.

Exercise 7. Compute $A - 5I_3$ and $(5I_3)A$, when

$$A = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix}$$

Solution. $A - 5I_3 = \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 & 3 \\ -8 & 2 & -3 \\ -4 & 1 & 3 \end{bmatrix}$

$(5I_3)A = 5\,(I_3A) = 5A = 5\begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -15 \\ -20 & 5 & 40 \end{bmatrix}$, or

$(5I_3)A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 9 & -1 & 3 \\ -8 & 7 & -3 \\ -4 & 1 & 8 \end{bmatrix} = \begin{bmatrix} 5\cdot 9 + 0 + 0 & 5(-1) + 0 + 0 & 5\cdot 3 + 0 + 0 \\ 0 + 5(-8) + 0 & 0 + 5\cdot 7 + 0 & 0 + 5(-3) + 0 \\ 0 + 0 + 5(-4) & 0 + 0 + 5\cdot 1 & 0 + 0 + 5\cdot 8 \end{bmatrix}$

$= \begin{bmatrix} 45 & -5 & 15 \\ -40 & 35 & -15 \\ -20 & 5 & 40 \end{bmatrix}$